ANALYSIS QUALIFYING EXAM
COMPLEX ANALYSIS
MAY 2017

NUMBER:____________

Let $\mathbb{C}$ denote the complex plane, let $\mathbb{D}$ denote the unit disk ($\mathbb{D} = \{z \mid |z| < 1\}$), and let $\mathbb{H}$ denote the upper half plane ($\mathbb{H} = \{z \mid \operatorname{Im} z > 0\}$.)

1) Let G be a simply connected domain and let $f : G \to G$ be a holomorphic map. Suppose that $f(z_0) = z_0$ for some $z_0 \in G$. Prove that

$$|f'(z_0)| \le 1.$$

2) Let $S = \{z \mid 0 < \operatorname{Im} z < \pi\}$. Find every univalent, conformal map ϕ that maps S onto $\mathbb{D}$ and satisfies $\phi(\pi/2) = 0$.

3) Evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + x + 1} dx.$$

(Note that $x^2 + x + 1 \ge 3/4$ on $\mathbb{R}$, so the integral is convergent.)

4) Let $S = \{re^{-i\pi r} \mid 0 \le r < \infty\}$. Prove that there is a branch L of $\log z$ in $\Omega = \mathbb{C} \setminus S$. Note that every odd integer is an element of Ω. What are the possible values of $L(1)$? Suppose that $L(1) = 0$, what is the value of $L(2n + 1)$ for each $n \in \mathbb{Z}$?

5) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Suppose that

$$\limsup_{z \to \infty} \frac{|f(z)|}{\ln |z|} < \infty.$$

What can you say about f?